A FINITE ELEMENT-GENERALIZED NETWORK ANALYSIS OF FINITE THICKNESS PHOTONIC CRYSTALS

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ABSTRACT

A Generalized Network Formulation for the analysis electromagnetic diffraction from a photonic crystal is proposed. The Finite Element method is employed to compute the impedance matrix which characterizes propagation through the unit cell. From this matrix, the bulk dispersion characteristic for the infinitely extending crystal, as well as the transmission and reflection coefficients for the case of a finite thickness sample of the crystal can be derived.

INTRODUCTION

Three-dimensional periodic structures, often referred to as photonic crystals borrowing the optics jargon, have been recently proved to have interesting characteristics, not yet available with the ordinary materials. They may act on electromagnetic waves in a similar way as natural crystals act on electron waves. If properly designed, these artificial materials can significantly changes electromagnetic waves propagation: entire frequency bands can be forbidden, and local modes can be trapped around local defects into the crystal lattice [1].

Possible applications of photonic crystals in the microwave and millimeter wave region of the electromagnetic wave spectrum range from circuits to antenna and stealth technology. For instance, they could be used to realized single structure multi-channel filters, low-loss guides for long antennas, quieter oscillator, polarizers and housing for quasi-optical applications, as well as to enhance planar antenna performances and reduce aperture antennas cross-talk. Some of these applications have indeed already been investigated [2] proving the usefulness of photonic crystals.

Much of the research in this area has been done experimentally and the need for flexible and efficient numerical tools is apparent. Only recently this aspect has been addressed and some well established methodology have been adapted to the periodic geometry and applied to the analysis of photonic crystals. Among these, the plane wave expansion method [3] and the Finite Difference both in Frequency Domain (FDFD) [4] and Time Domain (FDTD) [5,6] have been applied.

The plane wave expansion method was proposed first to compute the dispersion curve of electromagnetic waves in photonic crystals. It provides good results but may exhibits convergence problems when there is a large contrast in the dielectric constant values of the materials comprising the structure and cannot deal with the presence of conductors into the crystal. Although this limitation is not so stringent at optical frequency where dielectric materials are usually employed, this is not the case in the microwave and millimeter frequency ranges, where metallic and metallo-dielectric crystals with wide band-gap have been reported [5,7].

The method proposed by Pendry [4] to analyze this latter family of crystals, has proved effective. Such method is based on the usage of the FDFD algorithm to compute the transfer matrix through a thin, with respect to wavelength, slab of the crystal, assumed uniform along the direction of propagation. The transfer matrix so computed can be used to evaluate bulk dispersion as well as transmission and reflection through a finite thickness slab of material.

FDTD is most useful to deal with transmission through photonic crystals comprising non-linear material[6], even though it can be computationally demanding in the case of transmission through a thick slab. FDTD application to computation of dispersion

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diagram is also feasible, but requires to accurately chose the excitation to find all modes.

Finite Element Method (FEM) [8] has been also applied to the analysis of bulk dispersion of photonic crystals [9,10]. The method allows treating metal-dielectric crystals and avoids the staircase approximation of FD algorithms. However, the formulation previously adopted was based on the assumption of an infinitely extending three-dimensional periodic structure and thus cannot deal with transmission through a finite thickness crystal.

A new formulation is proposed here, in some respect similar to that developed by Pendry [5], which allows computing transmission and reflection through a finite thickness sample of material as well as the dispersion diagram of an infinitely extending crystal. It makes use of the FEM to evaluate the generalized impedance matrix which relates the electric and magnetic fields at two opposite sides of a unit cell of the crystal, in the hypothesis it infinitely extends only in two directions. Transmission through a finite thickness crystal, built by staggering a certain number of slabs, can be computed by cascading the impedance matrices of each slab.

THE FINITE ELEMENT - GENERALIZED NETWORK FORMULATION

Let us consider a thick planar structure extending among the planes $z=0$ and $z=d$, with two axes of periodicity as sketched in Fig. 1. This can be thought of as a slab of a three-dimensional periodic structure which can be obtained by staggering an arbitrary number of similar slabs into the *z* direction. Resorting to the Bloch's theorem, only a single cell Ω of the photonic crystal, delimited by the surface S, need to be analyzed. The fields inside Ω is expressed as:

$$\left[\mathbf{E}(\mathbf{r}),\mathbf{H}(\mathbf{r})\right]=\left[\mathbf{E}_p(\mathbf{r}),\mathbf{H}_p(\mathbf{r})\right]\exp(-j\mathbf{k}_t\cdot\mathbf{r}) \qquad (1)$$

where $\mathbf{k}_t$ is the transverse (with respect to the *z*-axis) propagation vector inside the material, and $\mathbf{E}_p(\mathbf{r})$, $\mathbf{H}_p(\mathbf{r})$ are the periodic part of the electromagnetic field, that is

$$\left[\mathbf{E}_p(\mathbf{r}+\mathbf{R}),\mathbf{H}_p(\mathbf{r}+\mathbf{R})\right]=\left[\mathbf{E}_p(\mathbf{r}),\mathbf{H}_p(\mathbf{r})\right] \qquad (2)$$

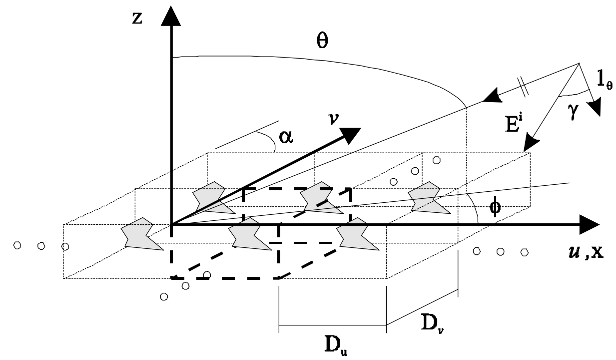

Fig. 1 - Geometry of a planar structure with periodicity in two directions.

where the vector $\mathbf{R}$ is a linear combination (with integer coefficients) of the primitive vectors $D_u\mathbf{u}$, $D_v\boldsymbol{v}$ denoting the spatial periodicity of the crystal in the plane *x*-*y*. Starting from the vector Helmholtz equation and applying the weighted residual procedure with $\mathbf{W}(\mathbf{r})=\mathbf{W}_p(\mathbf{r})\exp(-j\mathbf{k}_t\cdot\mathbf{r})$ as vector weighting functions, yields the weak form of the vector Helmholtz equation

$$\int_\Omega \Big(\left(\nabla+j\mathbf{k}_t\right)\times\mathbf{W}_p\cdot p^{-1}\left(\nabla-j\mathbf{k}_t\right)\times\mathbf{E}_p-k_0^2q\;\mathbf{W}_p\cdot\mathbf{E}_p\Big)d\Omega+jk_0Z_0\oint_{S_T\cup S_B}\left(\mathbf{W}_p\times\mathbf{H}_p\right)\cdot\mathbf{1}_n\,dS=0 \qquad (3)$$

In equation (3), $p=\mu_r$ and $q=\varepsilon_r$ if $\mathbf{F}_p=\mathbf{E}_p$, or $p=\varepsilon_r$ and $q=\mu_r$ if $\mathbf{F}_p=\mathbf{H}_p$. S_T and S_B denotes the top and bottom parts of the surface *S* delimiting the unit cell. At opposite sides of the unit cell which connects the cell analyzed with other cells of the structure, all the quantities are equal, except for the outward unit normal vector $\mathbf{1}_n$, which is opposite. Thus, contribution to the surface integral in the left hand side of equation (3) coming from such surfaces cancel out, and only contributions coming from the top $(z=0)$ and bottom $(z=-L)$ surfaces must be computed. For sake of clarity, in the following we will focus on the case and $\mathbf{F}_p=\mathbf{E}_p$, $p=\mu_r$, $q=\varepsilon_r$. By discretizing the unknown electric and magnetic fields with finite elements it is possible to derive a matricial relation between the electric and magnetic field associated to the edges lying on the surfaces S_T and S_B. Actually, all choices corresponding to different generalized network representations of the electromagnetic wave propagation through the slab are possible. For instance, by expressing the electric field at the top and bottom surfaces ($\mathbf{E}_T$ and $\mathbf{E}_B$, respectively) as a function of the magnetic field at the same surfaces ($\mathbf{H}_T$ and $\mathbf{H}_B$), the generalized impedance matrix for the propagation through the unit cell is built:

$$\begin{bmatrix} \mathbf{E}_T \\ \mathbf{E}_B \end{bmatrix} = \begin{bmatrix} \mathbf{Z}_{TT}(\mathbf{k}_t,\omega) & \mathbf{Z}_{TB}(\mathbf{k}_t,\omega) \\ \mathbf{Z}_{BT}(\mathbf{k}_t,\omega) & \mathbf{Z}_{BB}(\mathbf{k}_t,\omega) \end{bmatrix} \cdot \begin{bmatrix} \mathbf{H}_T \\ \mathbf{H}_B \end{bmatrix} \qquad (4)$$

This matrix can be used to evaluate both the dispersion relation of the photonic media and the transmission through and reflection by a photonic crystal of arbitrary thickness built by staggering a certain number of slabs.

Reflection and Transmission Coefficients

When an arbitrarily polarized plane wave impinges the slab coming from the half-space $z > 0$ as shown in Fig. 1, from the definition of impedance matrix and the boundary conditions, the equations (4) become

$$\begin{bmatrix} \mathbf{E}^{inc} + \mathbf{E}^{r} \\ \mathbf{E}^{t} \end{bmatrix} = \begin{bmatrix} \mathbf{Z}_{TT}(\mathbf{k}_t,\omega) & \mathbf{Z}_{TB}(\mathbf{k}_t,\omega) \\ \mathbf{Z}_{BT}(\mathbf{k}_t,\omega) & \mathbf{Z}_{BB}(\mathbf{k}_t,\omega) \end{bmatrix} \cdot \begin{bmatrix} \mathbf{H}^{inc} + \mathbf{H}^{r} \\ \mathbf{H}^{t} \end{bmatrix} \qquad (5)$$

where the superscripts *inc*, *r*, and *t* denote the incident, reflected, and transmitted fields, and the impedance matrix is computed for values of frequency and transverse propagation constant equal to those of the incident field. Reflected and transmitted electromagnetic fields may be expressed in terms of Bloch's waves with unknown amplitude coefficients denoted by the column vectors $\mathbf{E}^r_m, \mathbf{E}^t_m, \mathbf{H}^r_m, \mathbf{H}^t_m$. Maxwell's equations in the homogeneous media above and underneath the slab give the relation:

$$\begin{bmatrix} \mathbf{H}^{r}_{im} \\ \mathbf{H}^{t}_{mB} \end{bmatrix} = \begin{bmatrix} \mathbf{Z}^r_m & \mathbf{0} \\ \mathbf{0} & \mathbf{Z}^t_m \end{bmatrix} \cdot \begin{bmatrix} \mathbf{E}^r_m \\ \mathbf{E}^t_m \end{bmatrix} \qquad (6)$$

To derive a matrix equation relating the unknown amplitudes $\mathbf{E}^r_m, \mathbf{E}^t_m, \mathbf{H}^r_m, \mathbf{H}^t_m$ to the incident field, is necessary to introduce also the matrix $\mathbf{P}$ which projects the Bloch's functions onto the finite element basis used to derive the generalized impedance matrix of the slab. Then, introducing equation (6) into (5) yields:

$$\begin{bmatrix} \mathbf{P} - \mathbf{Z}_{TT}(\mathbf{k}_t,\omega)\mathbf{P}\mathbf{Z}^t_m & -\mathbf{Z}_{TB}(\mathbf{k}_t,\omega)\mathbf{P}\mathbf{Z}^r_m \\ -\mathbf{Z}_{BT}(\mathbf{k}_t,\omega)\mathbf{Z}^t_m & \mathbf{P} - \mathbf{Z}_{BB}(\mathbf{k}_t,\omega)\mathbf{P}\mathbf{Z}^r_m \end{bmatrix} \cdot \begin{bmatrix} \mathbf{E}^t_m \\ \mathbf{E}^r_m \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{Z}_{TB} \\ -\mathbf{I} & \mathbf{Z}_{BB} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{E}^{inc} \\ \mathbf{H}^{inc} \end{bmatrix} \qquad (7)$$

which can be solved for the unknown coefficient of the reflected and scattered field from a photonic crystal of finite thickness.

Dispersion Relation

Assuming an infinitely extending crystal, comprised of identical slab of thickness L staggered into the z direction, and invoking the Bloch's theorem, the fields at $z = 0$ and $z = -L$ must be equal except for the phase shift $\exp(jk_z L)$. From the generalized impedance matrix, we can derive the transmission matrix and solve the standard eigenvalue problem:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{E}_B \\ \mathbf{H}_B \end{bmatrix} = \exp(jk_z L) \cdot \begin{bmatrix} \mathbf{E}_B \\ \mathbf{H}_B \end{bmatrix} \qquad (8)$$

to find the propagation constant in the z direction.

RESULTS

Some results relative to simple crystals are shown next. Fig. 2 shows the gap in the transmission coefficient of a TM_z polarized incident wave through a two-dimensional photonic crystal obtained by staggering seven slabs comprised of a grating of dielectric rods with ε_r=2.98 radius r=0.37mm and period p=1.87mm. The slabs are staggered so as to form a square lattice. The results compare within a few percent with those presented in [11] and obtained using a technique similar to that proposed by Pendry. This simple two-dimensional structure exhibits a sharp gap into the transmission coefficient in the frequency range $0.322 \le a/\lambda_0 \le 0.440$.

Fig. 3 shows the dispersion diagram of the lowest TM_z polarized modes into a dielectric crystal comprised of circular dielectric rods with ε_r=8.9. The rods are located on a square lattice and have a radius

r=0.2a with a lattice constant. For this configuration and polarization, the forbidden range of frequency spans the values $0.322 \le a/\lambda_0 \le 0.440$. The dispersion diagram has been obtained by selecting the eigenvalue of equation (8) with unitary module. The method still present some numerical difficulties related to ill-conditioning of the generalized impedance matrix and this issue is currently under investigation.

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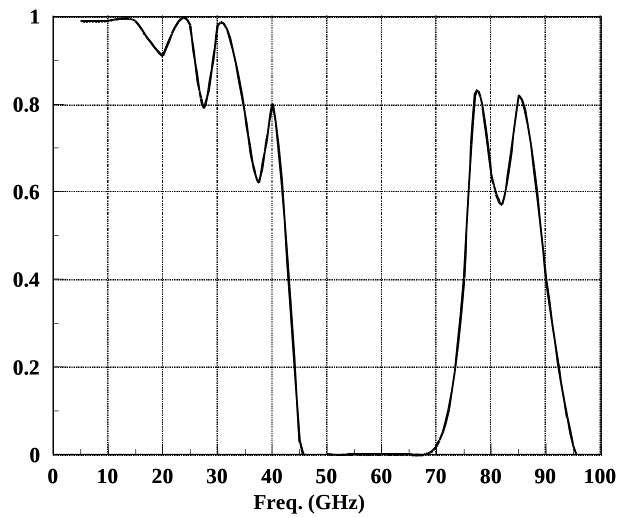

Fig. 2. Transmission coefficient through seven slab of dielectric rod gratings for a TM_Z polarized incident field. ε_r=2.98, r =0.37mm, a=1.87mm.

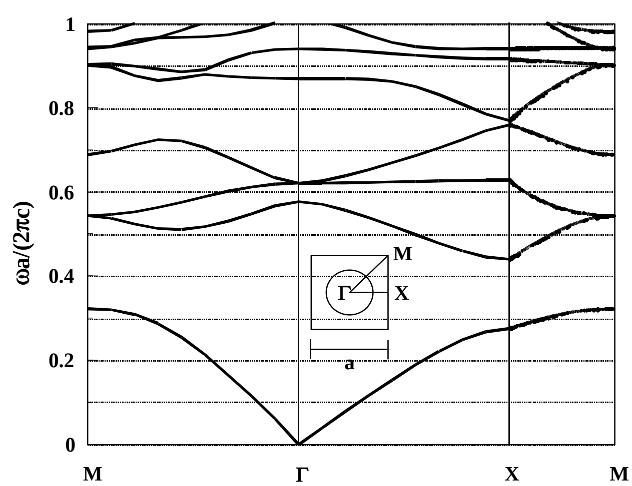

Fig. 3. Dispersion relation for TM_Z polarization in a square lattice of dielectric circular rods with ε_r=8.9 and r/a=0.2.